

A spectral Erdős-Stone-Bollobás theorem

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Abstract

Let $r \geq 3$ and $(c/r^r)^r \ln n \geq 1$. If G is a graph of order n and its largest eigenvalue $\mu(G)$ satisfies

$$\mu(G) \geq \left(1 - \frac{1}{r-1} + c\right)n,$$

then G contains a complete r -partite subgraph with $r-1$ parts of size $\lfloor (c/r^r)^r \ln n \rfloor$ and one part of size greater than $n^{1-c^{r-1}}$.

This result implies the Erdős-Stone-Bollobás theorem, the essential quantitative form of the Erdős-Stone theorem.

Moreover, if F is a fixed graph with chromatic number r , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \max \{ \mu(G) : G \text{ is of order } n \text{ and } F \not\subseteq G \} = 1 - \frac{1}{r-1}.$$

This result implies the Erdős-Stone-Simonovits theorem.

Keywords: *largest eigenvalue; r -partite subgraph; Erdős-Stone-Bollobás theorem; Erdős-Stone-Simonovits theorem*

This note is part of an ongoing project aiming to build extremal graph theory on spectral grounds, see, e.g., [6], [14, 22].

The fundamental Erdős-Stone theorem [9] states that, given $r \geq 3$ and $c > 0$, every graph with n vertices and $\lceil (1 - 1/(r-1) + c)n^2/2 \rceil$ edges contains a complete r -partite graph with each part of size $g(n, r, c)$, where $g(n, r, c)$ tends to infinity with n . In [4] Bollobás and Erdős found that $g(r, c, n) = \Theta(\log n)$, and in [3], [5], [8], and [12] the function $g(r, c, n)$ was determined with great precision.

Here we deduce the Erdős-Stone-Bollobás result from a weaker, spectral condition.

Our notation follows [2]. Let $K_r(s_1, \dots, s_r)$ be the complete r -partite graph with parts of size s_1, \dots, s_r , and let $\mu(G)$ be the largest adjacency eigenvalue of a graph G . Our main result:

Theorem 1 *Let $r \geq 3$, $(c/r^r)^r \ln n \geq 1$, and G be a graph with n vertices. If*

$$\mu(G) \geq \left(1 - \frac{1}{r-1} + c\right)n,$$

then G contains a $K_r(s, \dots, s, t)$ with $s \geq \lfloor (c/r^r)^r \ln n \rfloor$ and $t > n^{1-c^{r-1}}$.

As an easy consequence, we strengthen the Erdős-Stone-Simonovits theorem [10].

Theorem 2 *Let $r \geq 3$ and F be a fixed graph with chromatic number r . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \max \{ \mu(G) : G \text{ is of order } n \text{ and } F \not\subseteq G \} = 1 - \frac{1}{r-1}.$$

Remarks

- Since $\mu(G)$ is at least the average degree of G , Theorem 1 implies the following form of the Erdős-Stone-Bollobás theorem:

Let $r \geq 2$, $(c/r^r)^r \ln n \geq 1$, and G be a graph with n vertices. If G has $\lceil (1 - 1/(r-1) + c) n^2/2 \rceil$ edges, then G contains a $K_r(s, \dots, s, t)$ with $s \geq \lfloor (c/r^r)^r \ln n \rfloor$ and $t > n^{1-c^{r-1}}$.

This is slightly stronger than the result in [3] and is comparable to the results in [5].

- The relation between c and n in Theorem 1 needs explanation. First, for fixed c , it shows how large must be n to get a valid conclusion. But, in fact, the relation is subtler, for c itself may depend on n , e.g., letting $c = 1/\ln \ln n$, the conclusion is meaningful for sufficiently large n .
- Using random graphs, we see that almost all graphs on n vertices contain no $K_2(s, s)$ with s larger than $C \log n$ for some $C > 0$, independent of n . Hence, for constant c , Theorem 1 is essentially best possible.
- After this note has been made public, we were informed that Babai and Guiduli [11] have proved Theorem 2 using the Szemerédi Regularity Lemma; for a recent account on this matter see [1].

Proofs

A word about our proof methods seems in place. In an ongoing series of papers, e.g., [6, 7], [14, 22], we are developing a set of wide-range tools for use in extremal and spectral graph theory. Sometimes, as seen below, these tools are so effective that the proofs become vanishingly short.

Write $k_r(G)$ for the number of r -cliques of a graph G . The following facts play crucial roles in our proof of Theorem 1.

Fact 3 ([6], Theorem 2) *If $r \geq 2$ and G is a graph of order n , then*

$$k_r(G) \geq \left(\frac{\mu(G)}{n} - 1 + \frac{1}{r} \right) \frac{r(r-1)}{r+1} \left(\frac{n}{r} \right)^r.$$

□

Fact 4 ([13], Theorem 1) *Let $r \geq 2$, $c^r \ln n \geq 1$, and G be a graph of order n . If $k_r(G) \geq cn^r$, then G contains a $K_r(s, \dots, s, t)$ with $s = \lfloor c^r \ln n \rfloor$ and $t > n^{1-c^{r-1}}$.* □

Fact 5 *The number of edges of $T_r(n)$ satisfies $2e(T_r(n)) \geq (1 - 1/r)n^2 - r/4$.* \square

Proof of Theorem 1 In view of $\mu(G) \geq (1 - 1/(r-1) + c)n$, Fact 3 implies that

$$k_r(G) > c \frac{r-2}{r^r} n^r \geq \frac{c}{r^r} n^r.$$

Hence, Fact 4 implies that G contains a $K_r(s, \dots, s, t)$ with

$$s \geq (c/r^r)^r \ln n, \quad t > n^{1-c^{r-1}},$$

completing the proof. \square

Proof of Theorem 2 Theorem 1 implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \max \{ \mu(G) : G \text{ is of order } n \text{ and } F \not\subseteq G \} \leq 1 - \frac{1}{r-1}.$$

On the other hand, writing $T_s(n)$ for the s -partite Turán graph of order n , in view of Fact 5, we see that

$$\frac{\mu(T_{r-1}(n))}{n} \geq \frac{2e(T_{r-1}(n))}{n^2} \geq 1 - \frac{1}{r-1} - \frac{r-1}{4n^2}.$$

Since $T_{r-1}(n)$ is $(r-1)$ -partite, it contains no copy of F . Therefore,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \max \{ \mu(G) : G \text{ is of order } n \text{ and } F \not\subseteq G \} \geq 1 - \frac{1}{r-1},$$

completing the proof. \square

Concluding remark

Finally, a note about the project mentioned in the introduction: in this project we aim to give wide-range results that can be used further, adding more integrity to spectral extremal graph theory.

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